

Math 4200

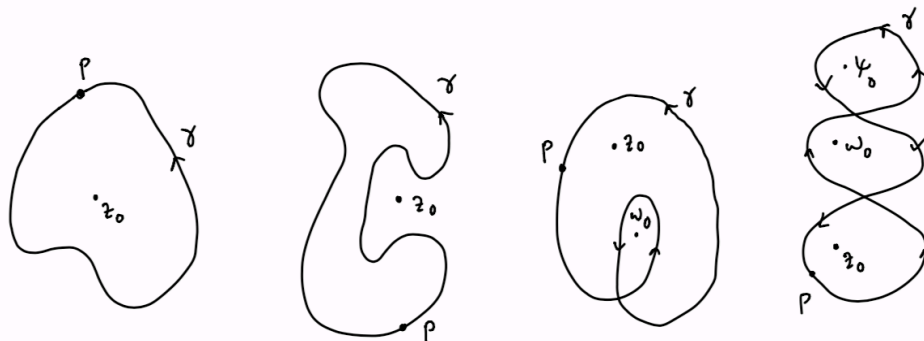
Monday October 5

2.4 Winding number (aka index), and Cauchy's integral formula

Announcements:

2.4 This section is about the magic fact that if a piecewise C^1 closed contour γ is given; and if $f(z)$ is analytic in an open simply-connected domain A containing γ , and if z_0 is "inside" γ , then $f(z_0)$ can be computed with an appropriate contour integral around γ . This is the *Cauchy Integral Formula* and is the basis for many amazing facts about analytic functions, and corollaries important in diverse pure and applied mathematics applications.

Step 1 What does it mean for z_0 to be "inside" γ ?



Def If γ is a continuous closed path in \mathbb{C} , $\gamma: [a, b] \rightarrow \mathbb{C}$, $\gamma(a) = \gamma(b) = P$, and if $z_0 \notin \gamma([a, b])$ then the *winding number* of γ about z_0 , also called the *index* of γ relative to z_0 , is how many times γ winds around z_0 in the counterclockwise direction. We write $I(\gamma; z_0)$ for this integer.

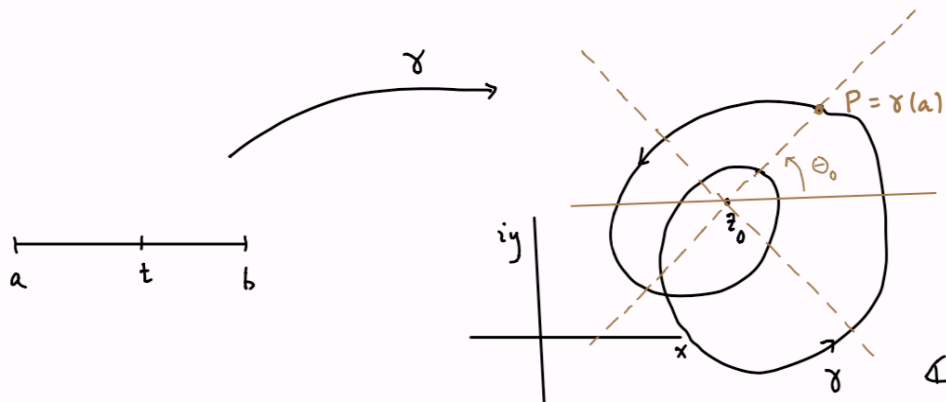
This number is usually easy to compute if you can see the image curve γ .

Examples: deduce the winding numbers of the various closed curves above, about the indicated points.

Def We say that z_0 is *inside* γ iff $I(\gamma; z_0) \neq 0$.

Lemma $I(\gamma; z_0)$ is a well defined integer for any continuous closed curve γ with $z_0 \notin \gamma([a, b])$.

proof:



$\gamma([a, b])$ is compact and contains its limit points, so $z_0 \notin \gamma([a, b])$ means γ stays a uniform distance away from z_0 , $|\gamma(t) - z_0| \geq \epsilon > 0$.

Claim (1): Make a choice θ_0 for the argument of $\gamma(a) - z_0$ (determined up to a multiple of 2π). Then there is a unique way to extend $\theta = \theta(t) = \arg(\gamma(t) - z_0)$ as a continuous function on the interval $[a, b]$, i.e. so that

$$\gamma(t) - z_0 = |\gamma(t) - z_0| e^{i\theta(t)} \quad \forall t \in [a, b].$$

proof: Consider the open half plane half plane indicated above:

$$H_1 = \left\{ z \in \mathbb{C} \mid \theta_0 - \frac{\pi}{2} < \arg(z - z_0) < \theta_0 + \frac{\pi}{2} \right\}.$$

As long as $\gamma([a, t]) \subseteq H_1$ there is a unique way to define

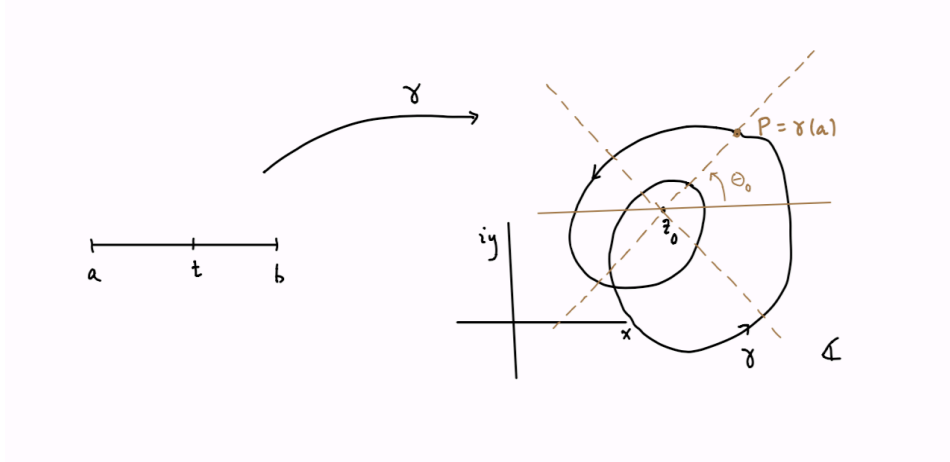
$$\theta(t) = \arg(\gamma(t) - z_0)$$

continuously, namely by requiring $\theta_0 - \frac{\pi}{2} < \theta(t) < \theta_0 + \frac{\pi}{2}$ ($\theta(t)$.) Let t_1 be the first $t > a$ with

$$\theta_0 - \frac{\pi}{2} = \arg(\gamma(t_1) - z_0) \quad \text{or} \quad \theta_0 + \frac{\pi}{2} = \arg(\gamma(t_1) - z_0).$$

Then extend $\theta(t)$ for $t > t_1$ using the neighbor halfplane

$$H_2 = \left\{ z \in \mathbb{C} \mid \arg(\gamma(t_1) - z_0) - \frac{\pi}{2} < \arg(z - z_0) < \arg(\gamma(t_1) - z_0) + \frac{\pi}{2} \right\}.$$



Continue inductively, finding t_2, t_3, \dots and half planes H_3, H_4, \dots if necessary.

Because $\gamma(t)$ is uniformly continuous and because $|\gamma(t) - z_0| \geq r$, this process terminates after a finite number of steps with

$$\theta(t) = \arg(\gamma(t) - z_0)$$

defined and continuous on the entire interval $[a, b]$, and so that

$$\gamma(t) - z_0 = |\gamma(t) - z_0| e^{i\theta(t)}.$$

(2) Define $I(\gamma; z_0) := \frac{1}{2\pi} (\theta(b) - \theta(a))$. Since $\gamma(a) = \gamma(b)$ and $\theta(b)$ and

$\theta(a)$ are argument choices of $\gamma(a) - z_0$, the index is an integer. Any other continuous construction of the argument function, say $\theta_1(t)$ would have

$$\theta_1(t) - \theta(t) = 2\pi k(t), \quad k(t) \in \mathbb{Z}.$$

Since $k(t)$ is a difference of continuous functions it is continuous on $[a, b]$ and since it only takes on integer values, i.e. $\theta_1(t) = \theta(t) + 2\pi k$, it must be constant. Thus

$$\frac{1}{2\pi} (\theta_1(b) - \theta_1(a)) = \frac{1}{2\pi} (\theta(b) - \theta(a))$$

so $I(\gamma; z_0)$ is well defined.

Theorem If γ, z_0 are as in the preceding discussion, and if γ is also piecewise C^1 , then index can be computed with a contour integral:

$$I(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz.$$

motivation: locally, in polar coordinates, $z = z_0 + r e^{i\theta}$

$$z = z_0 + r e^{i\theta}$$

$$dz = (dr)e^{i\theta} + r e^{i\theta} i d\theta$$

$$\Rightarrow \frac{dz}{z - z_0} = \frac{(dr)e^{i\theta} + r e^{i\theta} i d\theta}{r e^{i\theta}} = \frac{dr}{r} + i d\theta.$$

proof: Let $a \leq s \leq b$, $\theta_0, \theta(t), H_1, t_1, H_2, t_2 \dots$ as in the earlier discussion. If γ is C^1 , then for $a \leq s \leq t_1$ and using polar coordinates in the initial half plane H_1 ,

$$\gamma(t) = z_0 + r(t)e^{i\theta(t)}$$

$$\begin{aligned} \frac{1}{2\pi i} \int_a^s \frac{1}{\gamma(t) - z_0} \gamma'(t) dt &= \frac{1}{2\pi i} \int_a^s \frac{r'(t)e^{i\theta(t)} + r(t)i e^{i\theta(t)}\theta'(t)}{r(t)e^{i\theta(t)}} dt \\ &= \frac{1}{2\pi i} \left(\ln(r(s)) - \ln(r(a)) + i(\theta(s) - \theta(a)) \right). \end{aligned}$$

Continue for $t_1 \leq s \leq t_2, \dots$ and adding more subintervals if γ is only piecewise C^1 .

Using telescoping series, deduce that for all $a \leq s \leq b$,

$$\frac{1}{2\pi i} \int_a^s \frac{1}{\gamma(t) - z_0} \gamma'(t) dt = \frac{1}{2\pi i} \left(\ln\left(\frac{r(s)}{r(a)}\right) + i(\theta(s) - \theta(a)) \right).$$

At $s = b$ for this closed curve $\ln\left(\frac{r(b)}{r(a)}\right) = 0$ so

$$\frac{1}{2\pi i} \int_a^s \frac{1}{\gamma(t) - z_0} \gamma'(t) dt = \frac{1}{2\pi} (\theta(b) - \theta(a)) = I(\gamma; z_0).$$

Q.E.D.

Examples

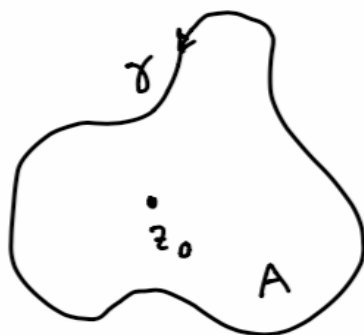
a) Show that for $\gamma_n(t) = z_0 + r e^{i t}$, $0 \leq t \leq 2 n \pi$, $n \in \mathbb{Z}$ that the winding number of n agrees the contour integral formula

$$I(\gamma; z_0) = \frac{1}{2 \pi i} \int_{\gamma} \frac{1}{z - z_0} dz.$$

b) Let A be an open connected domain with boundary a p.w. C^1 simple closed curve γ oriented counterclockwise, $z_0 \in A$. Show

$$I(\gamma; z_0) = 1$$

via contour replacement.



Remark If γ is any simple closed curve in $\mathbb{C} \setminus \{0\}$ then it turns out that γ is homotopic as closed curves to one of the γ_n 's in part (a). And so one could use the deformation theorem to deduce its index. This homotopy fact is typically proven in courses on algebraic topology where one studies the *fundamental group* of various spaces, including $\mathbb{C} \setminus \{0\}$ and the unit circle.

The Cauchy Integral Formula (which we will see is amazing in its consequences):

Let $A \subseteq \mathbb{C}$ be open and connected, $f: A \rightarrow \mathbb{C}$ analytic;

$\gamma: [a, b] \rightarrow \mathbb{C}$ a piecewise C^1 closed contour in A , $z_0 \notin \gamma([a, b])$.

Let γ be *contractible* in A , i.e. homotopic to a point as closed curves in A . (If A is simply connected this is automatic.)

Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz = f(z_0) I(\gamma; z_0).$$

(So, if z_0 is inside γ then $f(z_0)$ is determined and computable just from the values of f along γ !!!)

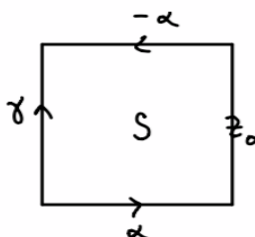
proof: Let

$$g(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

1) g is analytic in $A \setminus \{z_0\}$ and continuous at z_0 .

2) So the modified rectangle lemma holds in any subdisk of A containing z_0 (see last Friday notes) and the local antiderivative theorem holds for $g(z)$.

3) So we may apply the *homotopy lemma* with the contractible curve diagram below, because it only depends on local antiderivatives and a subdivision argument in the domain and range of the homotopy.



to deduce $\int_{\gamma} g(z) dz = 0$.

4) Since z_0 is not on the contour defined by γ ,

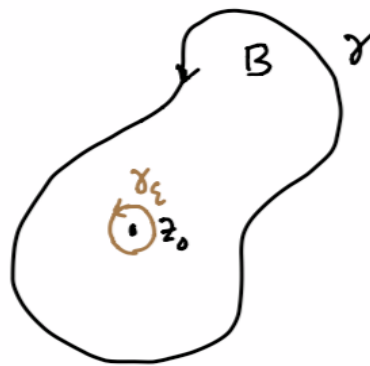
$$0 = \int_{\gamma} g(z) dz = \int_{\gamma} \frac{f(z) - f(z_0)}{z - z_0} dz = \int_{\gamma} \frac{f(z)}{z - z_0} dz - f(z_0) 2\pi i I(\gamma; z_0)$$

Q.E.D.

Remark: If γ is a counter-clockwise simple closed curve bounding a subdomain B in A , with z_0 inside γ , then we already checked that $I(\gamma; z_0) = 1$. So in this case the CIF reads

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$

One can prove this important special case of the Cauchy integral formula with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic. This is a post-exam homework problem.



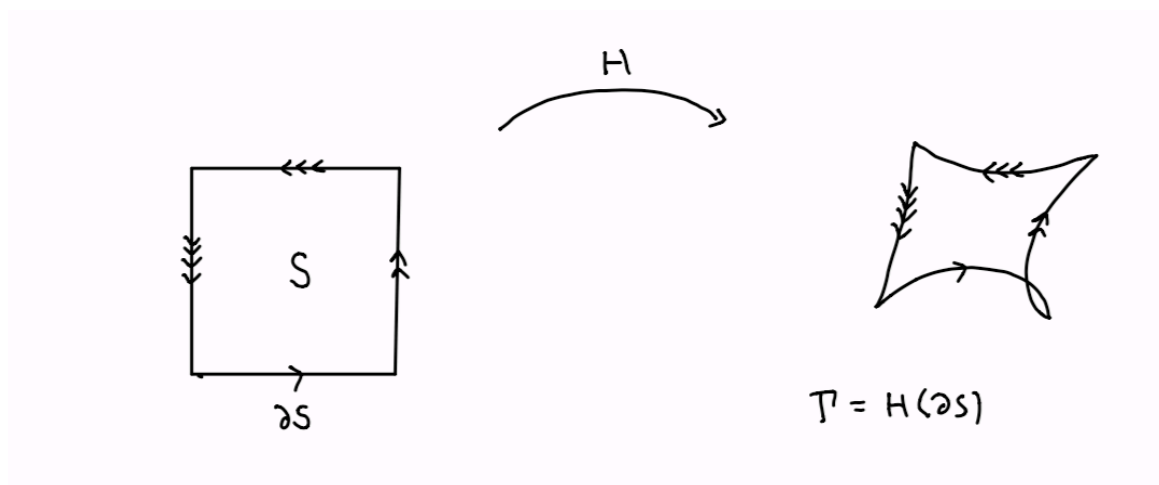
From Friday:

Homotopy Lemma Let $A \subseteq \mathbb{C}$ be open and connected. Let $f: A \rightarrow \mathbb{C}$ be analytic. Let

$$S = \{(s, t) \mid 0 \leq s \leq 1, 0 \leq t \leq 1\} \text{ and } \partial S$$

denote the unit square and its boundary, oriented counterclockwise. Let $H: S \rightarrow A$ be continuous, with $\Gamma := H(\partial S)$ a piecewise C^1 contour. Then

$$\int_{\Gamma} f(z) dz = 0.$$

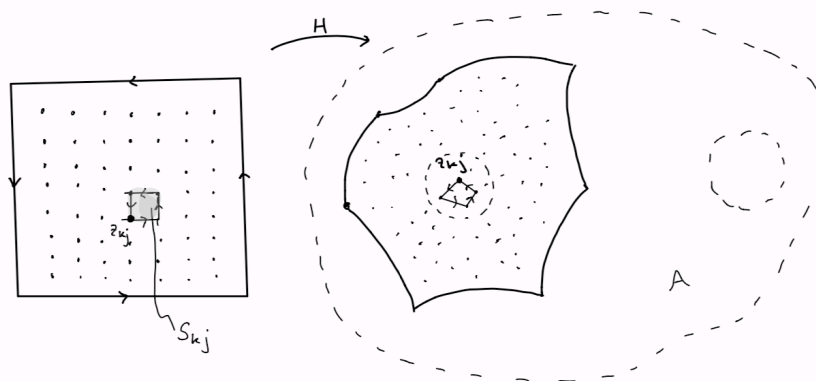


proof of the homotopy lemma: Subdivide S into n^2 subsquares of side lengths n^{-1} . The dots in the diagram on the left indicate their vertices. number the squares as you would a matrix, and let S_{kj} be a typical subsquare, with z_{kj} be the image under the homotopy of its lower left corner. Since H is continuous and S is compact, the image $H(S) \subseteq A$ is compact. Write

$$H(\delta S) = \Gamma$$

$$H(\delta S_{kj}) = \Gamma_{kj}.$$

Replace any of the four subarcs of each Γ_{kj} which are not C^1 with constant speed line segment paths between the image vertices.



By interior cancellation,

$$\int_{\Gamma} f(z) dz = \sum_{k,j} \int_{\Gamma_{kj}} f(z) dz.$$

Note:

1) $H(S)$ is compact, $H(S) \subseteq A$ open, so by the Positive Distance Lemma you're proving in this week's homework

$$\exists \varepsilon > 0 \text{ such that } \forall z \in H(S), D(z; \varepsilon) \subseteq A.$$

2) H is continuous on S so H is uniformly continuous. Thus for ε as in (1),

$$\exists \delta > 0 \text{ such that } \|(s, t) - (\tilde{s}, \tilde{t})\| < \delta \Rightarrow |H(s, t) - H(\tilde{s}, \tilde{t})| < \varepsilon.$$

3) If n is large enough so that the diagonal length of the subsquares is less than δ , then each

$$H(S_{kj}) \subseteq D(z_{kj}; \varepsilon) \subseteq A, z_{kj} = H(s_k, t_j).$$

4) By the local antidifferentiation theorem in $D(z_{kj}; \varepsilon)$, each

$$\int_{\Gamma_{kj}} f(z) dz = 0 \Rightarrow \int_{\Gamma} f(z) dz = 0. \quad \text{Q.E.D.!!!}$$